Analytical approaches to cellular automata for traffic flow: Approximations and exact solutions

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Abstract. Cellular automata have turned out to be important tools for the simulation of traffic flow. They are designed for an efficient impletmentation on the computer, but hard to treat analytically. Here we discuss several approaches for an analytical description of the Nagel-Schreckenberg (NaSch) model and its variants. These methods yield the exact solution for the special case $v_{max} = 1$ of the NaSch model and are good approximations for higher values of the velocity ($v_{max} > 1$). We discuss the validity of these approximations and the conclusions for the underlying physics that can be drawn from the success or failure of the approximation.

1 Introduction

Cellular automata (CA) do not only serve as simple model systems for the investigation of problems in statistical mechanics, but they also have numerous applications for 'real' problems [1]. Therefore it is not suprising that in recent years CA have become quite popular for the simulation of traffic flow (see e.g. [2]).

Here we do not want to discuss the question whether CA are suitable models for the description of traffic. We try to provide a tool box for the determination of the basic properties of the stationary state using analytical methods. These methods can also be applied to other CA models.

CA are – by design – ideal for large-scale computer simulations. On the other hand, analytical approaches for the description of CA are notoriously difficult. This is mainly due to the discreteness and the use of a parallel updating scheme (which introduces 'non-locality' into the dynamics). In addition, these models are defined through dynamical rules (e.g. transition probabilities) and usually one does not have a 'Hamiltonian' description. Therefore standard methods are not applicable. Furthermore, one has to deal with systems far from equilibrium which do not satisfy the detailed balance condition.

However, there is a need for exact solutions or, at least, for good approximations. These results as well as other exact statements may help to greatly reduce the need for computer resources. The interpretation of simulation data is often

difficult because of the 'numerical noise'. Even in the cases where an exact solution is not possible, a combination of analytical and numerical methods might provide better insights.

Here we will present analytical approaches which allow to solve the special case $v_{max} = 1$ of the Nagel-Schreckenberg model [3] exactly. For $v_{max} > 1$ these methods are only approximations, but they become exact in certain limits. We will discuss the quality of these approximations and how relevant information about the underlying physics can be obtained.

2 Nagel-Schreckenberg model

The Nagel-Schreckenberg (NaSch) model [3] is a probabilistic cellular automaton. Space and time are discrete and hence also the velocities. The road is divided into cells of length 7.5 m. Each cell can either be empty or occupied by just one car. The state of car j ($j = 1, \ldots, N$) is characterised by an internal parameter v_j ($v_j = 0, 1, \ldots, v_{max}$), the momentary velocity of the vehicle. In order to obtain the state of the system at time $t + 1$ from the state at time t, one has to apply the following four rules to all cars at the same time (parallel dynamics):

R1 Acceleration: $v_j \to v'_j = v_j + 1$ (only if $v_j < v_{max}$)

R2 Braking: $v_j \to v'_j = d_j$ (for $d_j > v_j$)

R3 Randomization: $v'_j \xrightarrow{p} v''_j = v'_j - 1$ with probability p (if $v'_j > 0$)

R4 Driving: car j moves v''_j cells.

Here d_j denotes the number of empty cells in front of car j, i.e. the gap or headway. One timestep $t \to t+1$ corresponds to approximately 1 sec in real time [3]. For simplicity we will only consider periodic boundary conditions so that the

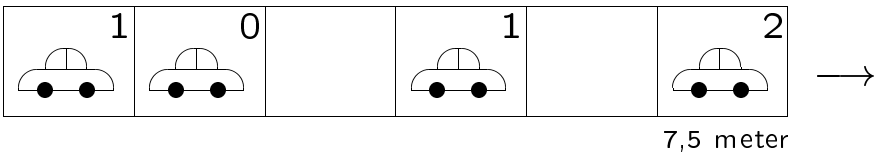

Fig. 1. Configuration in the Nagel-Schreckenberg model. The number in the upper right corner is the velocity of the car.

number of cars is conserved[1]. The maximum velocity v_{max} can be interpreted as a speed limit and is therefore taken to be identically for all cars.

[1] A discussion of the effects of open boundary conditions can be found in the contribution by N. Rajewsky [4]. J. Krug [5] and [6] investigate the effects of disorder.

The four steps have simple interpretations. Step R1 means that every driver wants to drive as fast as possible or allowed. Step R2 avoids crashes between the vehicles. The randomization step R3 takes into account several effects, e.g. road conditions (e.g. slope, weather) or psychological effects (e.g. velocity fluctuations in free traffic). An important consequence of this step is the introduction of an asymmetry between acceleration and deceleration, e.g. overreactions at braking which are important for the occurance of phantom traffic jams. Finally, step R4 is the actual motion of the vehicles.

The NaSch model is a minimal model in the sense that all four steps R1-R4 are necessary to reproduce the basic properties of real traffic. For more complex situations (e.g. 2-lane traffic) additional rules have to be formulated.

In the following sections several methods for an analytical description of the NaSch model will be presented.

3 Simple Mean-Field Theory

The simplest analytical approach to the NaSch model is a (microscopic) mean-field (MF) theory. Here one considers the density $c_v(j,t)$ of cars with velocity v at site j and time t. In the MF approach, correlations between sites are completely neglected.

For $v_{max} = 1$ the MF equations for the stationary state ($t \to \infty$) read [7]:

$$c_0 = (c + pd)c, \tag{1}$$

$$c_1 = qcd \tag{2}$$

with $d = 1 - c$ and $q = 1 - p$. The flow is simply given by $f(c) = c_1 = qc(1-c)$.

For random-sequential dynamics the MF approach is known to be exact for $v_{max} = 1$ [3]. For parallel dynamics, however, MF theory underestimates the flow considerably (see Fig. 2). This shows that correlations are important in this case. These correlations lead to an increase of the flow. Therefore one expects a particle-hole attraction (particle-particle repulsion), i.e. the probability to find an empty site in front of a car is enhanced compared to a completely random configuration. This picture will be confirmed by the exact solution.

The MF equations for arbitrary $v_{max} > 1$ are given in [7]. One finds the same qualitative behaviour, i.e. MF theory underestimates the flow even more than for $v_{max} = 1$.

4 Garden of Eden States

An important effect of the parallel dynamics is the existence of configurations which can not be reached dynamically [8]. These states are called Garden of Eden (GoE) states or paradisical states since they have no predecessor. An example for a GoE state is given in Fig. 3. Note that the velocity is equal to the number of cells that the car moved in the previous timestep. For the configuration shown in Fig. 3 this implies that the two cars must have occupied the same cell before

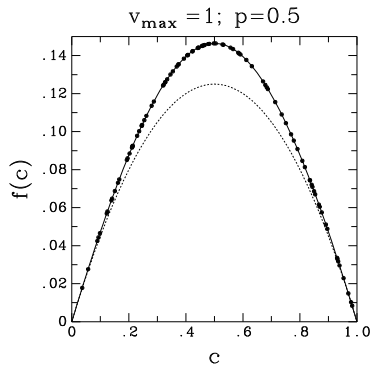

Fig. 2. Fundamental diagram for $v_{max} = 1$: Comparison of computer simulations (•) with the exact solution (continous line) and the mean-field result (broken line).

the last timestep[2]. Since this is forbidden in the NaSch model, the configuration shown can never be generated by the dynamics.

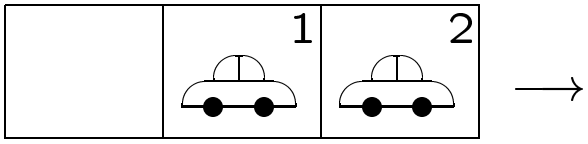

Fig. 3. A Garden of Eden state for the model with $v_{max} \geq 2$.

The simple mean-field theory presented in the previous section does not take into account the existence of GoE states. One can therefore hope that by eliminating all GoE states and applying mean-field theory in the reduced configuration space (paradisical mean-field, pMF) one will find a considerable improvement of the MF results.

For $v_{max} = 1$ all states containing the local configurations $(0, 1)$ or $(1, 1)$ are GoE states, i.e. the cell behind a car with velocity 1 must be empty. This only affects eq. (1) and the equations for pMF theory read:

$$c_0 = \mathcal{N}(c_0 + pd)c, \tag{3}$$

$$c_1 = \mathcal{N}qcd, \tag{4}$$

where the normalization $\mathcal{N}$ ensures $c_0 + c_1 = c$ and is given explicitly by $\mathcal{N} = \frac{1}{c_0+d}$.

[2] The reader should check whether Fig. 1 depicts a GoE state or not !

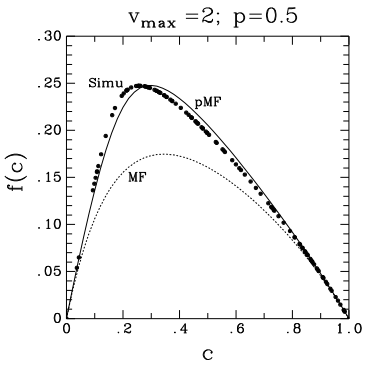

Fig. 4. Comparison of the fundamental diagrams obtain from MC simulations with the results of mean-field theory (MF) and mean-field theory without GoE states (paradisical mean-field, pMF).

Solving (4) for c_1 by using $c_0 = c - c_1$, one obtains c_1 as a function of the density c. Since the flow is given by $f(c) = c_1$,

$$f(c,p) = \frac{1}{2}\left(1 - \sqrt{1 - 4q(1-c)c}\right) \tag{5}$$

(with $q = 1 - p$), we recover the exact solution for the case $v_{max} = 1$ found first in [9] using the cluster approximation (see Section 5). Note that the flow depends on the density only via $c(1-c)$, reflecting the particle-hole symmetry of the NaSch model for $v_{max} = 1$.

For $v_{max} = 2$ one has to take into account more GoE states [8]. pMF is no longer exact, but it still leads to a considerable improvement of the MF results (see Fig. 4).

It is interesting to note that for random-sequential dynamics MF theory is exact whereas for parallel dynamics pMF is exact. This suggests that the important difference between random-sequential and parallel dynamics is the existence of Garden of Eden states in the latter. This is probably not only true for the NaSch model, but rather is a general property of CA models.

5 Cluster Approximation

The cluster approximation is a systematic improvement of MF theory which takes into account short-ranged correlations between the cells. In the n-cluster approximation a cluster of n neighbouring cells is treated exactly. The cluster is then coupled to the rest of the system in a self-consistent way. Related approximations have already been used (under different names) for other models [10,11,12,13].

In order to simplify the description, we choose a slightly different update-ordering R2-R3-R4-R1 instead of R1-R2-R3-R4, i.e. we look at the system after

the acceleration step. Then there are no cars with $v = 0$ and effectively we have to deal with one equation less. In the following we will use occupation variables n_j where $n_j = 0$, if cell j is empty, and $n_j = v$, if cell j is occupied by a car with velocity v.

If we denote the probability to find the system in a configuration $(n_1, \ldots, n_L)$ by $P(n_1, \ldots, n_L)$ the 1-cluster approximation means a simple factorization

$$P(n_1, \ldots, n_L) = \prod_{j=1}^{L} P(n_j). \tag{6}$$

This is nothing but the mean-field theory of section 3. For the 2-cluster approximation one has a factorization of the form

$$P(n_1, \ldots, n_L) \propto P(n_1, n_2)P(n_2, n_3) \cdots P(n_{L-1}, n_L)P(n_L, n_1). \tag{7}$$

The 3-cluster approximation is depicted graphically in Fig. 5.

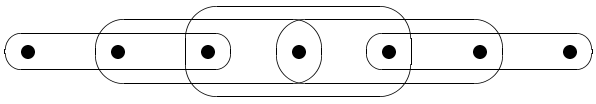

Fig. 5. The 3-cluster approximation. Shown are the central 3-cluster and all neighbouring 3-clusters with a nonvanishing overlap.

In general, the master equation leads to $(v_{max} + 1)^n$ nonlinear equations in n-cluster approximation. Although this number can be reduced by using certain consistency conditions [11], a solution is only feasible for relatively small cluster-sizes [7]. The quality of the approximation improves with increasing n and for $n \to \infty$ the n-cluster result becomes exact.

However, for $v_{max} = 1$ already the 2-cluster approximation is exact [7,9]. This is understandable from the results of the previous section since in this case the 2-cluster approximation is able to take into account all GoE states. The 2-cluster probablities for the stationary state are given explicitly by

$$\begin{aligned} P(0,0) &= 1 - c - P(1,0), \\ P(1,1) &= c - P(1,0), \\ P(1,0) &= P(0,1) = \frac{1}{2q}\left[1 - \sqrt{1 - 4qc(1-c)}\right], \end{aligned} \tag{8}$$

where again $q = 1 - p$. Using $f(c) = qP(1,0)$, one obtains the exact result for the flow (see (5)). Note that the particle-hole attraction is apparent from these results, since $P(0,1) \geq P(0)P(1) = c(1-c)$.

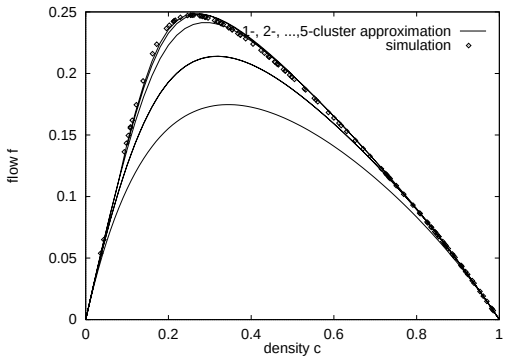

Fig. 6. Comparison of simulation results with the n-cluster approximation ($n = 1, \ldots, 5$) for the fundamental diagram for $v_{max} = 2$ and $p = 1/2$.

For $v_{max} = 2$ the fundamental diagrams obtained from the n-cluster approximation ($n = 1, \ldots, 5$) are compared in Fig. 6 with results of Monte Carlo simulations. One can see a rapid convergence, already for $n = 4$ the difference between the simulation and the cluster result is extremely small.

5.1 Correlation Length

As an application of the cluster approximation we will compute density-density correlations for $v_{max} = 1$ in the following. Using occupation numbers $n_j = 0, 1$ the density-density correlation function is defined by

$$\langle n_1 n_r \rangle = {\sum_{\{n_j\}}}' n_1 n_r P(n_1, \ldots, n_L) \tag{9}$$

where the prime indicates that the sum runs over all states with fixed particle number $N = n_1 + \cdots + n_L$.

In order to evaluate the sum in (9), it is convenient to use a grand-canonical description. One introduces a fugazity z which controls the average number of particles $\langle n_j \rangle$ and sums over all configurations in (9). Using the 2-cluster approximation (7) – which is exact for $v_{max} = 1$ – the correlation function is given by

$$\langle n_1 n_r \rangle = \frac{1}{Z_{gc}} \sum_{\{n_j\}} n_1 n_r z^N P(n_1, n_2) P(n_2, n_3) \cdots P(n_{L-1}, n_L) P(n_L, n_1) \tag{10}$$

with $N = \sum_{j=1}^{L} n_j$ and the normalization

$$Z_{gc} = \sum_{\{n_j\}} z^N \prod_{j=1}^{L} P(n_j, n_{j+1}). \tag{11}$$

Introducing the transfer matrix

$$\tilde{\mathbf{P}} = \begin{pmatrix} P(0,0) & \sqrt{z}P(0,1) \\ \sqrt{z}P(1,0) & zP(1,1) \end{pmatrix} \tag{12}$$

this can be written succinctly as

$$Z_{gc} = \mathrm{Tr}\ \tilde{\mathbf{P}}^L\ , \tag{13}$$

$$\langle n_1 n_r \rangle = \frac{1}{Z_{gc}} \mathrm{Tr} \left(\mathbf{Q}\tilde{\mathbf{P}}^{r-1}\mathbf{Q}\tilde{\mathbf{P}}^{L-r+1} \right). \tag{14}$$

with the matrix $Q(n_1, n_2) = n_1 \tilde{\mathbf{P}}(n_1, n_2)$.

The correlation length ξ can be obtained from the asymptotic behaviour ($r \to \infty$) of the correlation function

$$\langle n_1 n_r \rangle - c^2 \propto e^{-r/\xi} \tag{15}$$

where $c = \langle n_j \rangle$ is the (average) density of cars. ξ is determined by the ratio of the eigenvalues $\lambda_\pm$ of $\tilde{\mathbf{P}}$ (with $|\lambda_+| \geq |\lambda_-|$):

$$\xi^{-1} = \ln \left| \frac{\lambda_-}{\lambda_+} \right| \tag{16}$$

The explicit expression for ξ is rather lenghty. Therefore we just note that, for fixed p, ξ is maximal for $c = 1/2$. $\xi(c = 1/2)$ diverges only for $p \to 0$. In that case one finds that $\xi(c = 1/2) \propto p^{-1/2}$. Monte Carlo simulations show that the same behaviour still occurs (at density $c = \frac{1}{v_{max}+1}$) for $v_{max} > 1$ [14,15]. Therefore, the correlation function already gives an indication that the system is not critical for $p > 0$. The simulations show that there is no qualitative difference between the cases $v_{max} = 1$ and $v_{max} > 1$ as far as the phase transition is concerned [15].

The above results demonstrate that, although the 2-cluster approximation is exact, not all correlation functions are short-ranged or even of finite range.

5.2 Further Applications

We briefly mention other results which have been obtained analytically using the cluster approximation. In [16] the distributions of gaps and the distance between jams have been calculated for $v_{max} = 1$ using the 2-cluster approximation. For the gap distribution one recovers the (exact) result which will be derived in Sect. 6 (see eq. (18)). For the calculation of the distribution of gaps between jams one defines every vehicle with velocity 0 to be jammed. The distance between two jams is then given by the distance between a car with velocity 0 and the next car with velocity 0. In [17] the probability $\mathcal{P}(t)$ of a time headway t has been investigated. This quantity is defined in analogy to measurements on real traffic where a detector registers the time interval between the passing of consecutive cars.

6 Car-Oriented Mean-Field Theory

The car-oriented mean-field (COMF) theory [18] is another possibility to take into account correlations in an analytical description[3].

The central quantitiy in COMF is the probability $P_n(v)$ to find exactly n empty cells (i.e. a gap of size n) in front of a car with velocity v. In this way certain longer-ranged correlations are already taken into account. The essence of COMF is now to neglect correlations between the gaps.

For $v_{max} = 1$ the system of equations resulting from the master equation has the following form:

$$\begin{aligned} P_0(t+1) &= \bar{g}(t)\left[P_0(t) + qP_1(t)\right], \\ P_1(t+1) &= g(t)P_0(t) + \left[qg(t) + p\bar{g}(t)\right]P_1(t) + q\bar{g}(t)P_2(t), \\ P_n(t+1) &= pg(t)P_{n-1}(t) + \left[qg(t) + p\bar{g}(t)\right]P_n(t) + q\bar{g}(t)P_{n+1}(t), \quad (n \geq 2) \end{aligned} \tag{17}$$

where $g(t) = q\sum_{n\geq 1} P_n(t) = q[1 - P_0(t)]$ ($\bar{g}(t) = 1 - g(t)$) is the probability that a car moves (does not move) in the next timestep. As an example for the derivation of these equations we consider the equations for $n \geq 2$. Since the velocity difference of two cars is at most 1, a gap of n cells at time $t+1$ must have evolved form a gap of length $n-1$, n, or $n+1$ in the previous timestep. A headway of $n-1$ cells evolves into a headway of n cells only if the first car moves (with probability $g(t)$) and the second car brakes in the randomization step (probability p), i.e. the total probability for this process is $pg(t)P_{n-1}(t)$. Similarly, the headway remains constant only if either both cars move (probability $qg(t)$) or both cars do not move (probability $p\bar{g}(t)$). Finally, the headway is reduced by one, if only the second car moves (probability $q\bar{g}(t)$).

Although this infinite system of non-linear equations looks more difficult than those of the cluster approximation, a solution possible using generating functions [18]. For $v_{max} = 1$ one finds

$$\begin{aligned} P_0 &= \frac{1}{2qc}\left[2qc - 1 + \sqrt{1 - 4qc(1-c)}\right], \\ P_n &= \frac{P_0}{p}\left(\frac{p(1-P_0)}{P_0 + p(1-P_0)}\right)^n \quad (n \geq 1), \end{aligned} \tag{18}$$

and for the flow $f(c,p) = cg$ again the exact solution is reproduced.

For $v_{max} = 2$ one has two coupled systems of the type (18), since one has to distinguish $P_n(v=1)$ and $P_n(v=2)$ [18]. This system can also be solved, but as for the other methods, one does not find an exact solution.

In Fig. 7 results for the distribution of headways are shown. For $v_{max} = 1$, the distribution has just one maximum at $n = n_0$, with $n_0 = 1$ for small c and $n = 0$ for large c. For higher velocities, however, there is a density regime where the headway distribution exhibits two local maxima [16]. The maximum at $n = 0$ corresponds to jammed cars and the second maximum corresponds to free flowing cars.

[3] A similar method is discussed in [19].

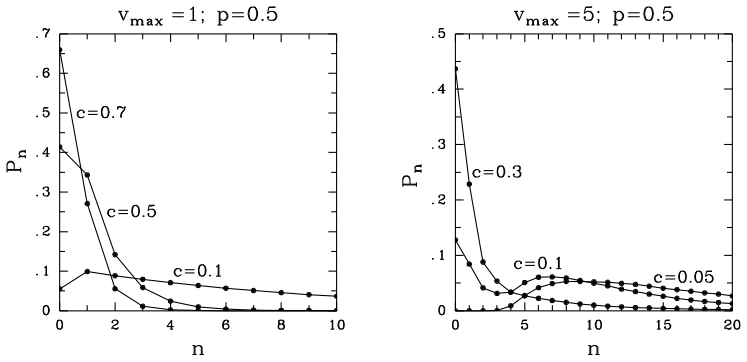

Fig. 7. Distribution of headways for different densities for $v_{max} = 1$ and $p = 0.5$ (left) and $v_{max} = 5$ and $p = 0.5$ (right).

7 Cluster Approximation vs. COMF

In this section the results of the cluster approximation and COMF in special limits are compared. As already mentioned, both methods yield the exact solution for $v_{max} = 1$. This is related to the fact that paradisical mean-field is exact in that case and both methods are able to identify all GoE states. For $v_{max} = 2$ the situation is different. Here all three methods are only approximative. The quality of COMF is usually between those of the 2-cluster and 3-cluster approximations.

In the limit $p \to 0$, COMF and the 3-cluster approximation become exact, in contrast to the 2-cluster approximation (see Fig. 8). Even for values of $p \approx 0.1$ there is an excellent agreement between the fundumental diagrams obtained analytically and the Monte Carlo simulations. Since 'realistic' values of p are in the region $p \sim 0.1 - 0.2$ the approximations are indeed applicable in the relevant parameter regime.

The limit $p \to 1$ is difficult to investigate numerically. Preliminary results show that COMF not exact in this limit. The cluster approximations give a much better agreement with Monte Carlo simulations, but it is not clear yet whether it becomes exact or not.

In order to understand the differences between COMF and the cluster approximation we calculate the distribution of jam sizes using these methods. Let C_n be the probability to find a (compact) jam of length n, i.e. n consecutive occupied cells. C_n is proportional to $P(\underline{0}|1)P(\underline{1}|1)\cdots P(\underline{1}|1)P(\underline{1}|1)P(\underline{1}|0)$ in 2-cluster approximation. Here we have again used the different update-ordering R2-R3-R4-R1. $P(\underline{\sigma}|\tau)$ denotes the conditional probability $P(\sigma,\tau)/(\sum_{\tilde{\tau}} P(\sigma,\tilde{\tau}))$. Note that the case $n = 1$ has to be treated separately, $C_1 \propto \sum_{v=1}^{v_{max}} P(\underline{0}|v)P(\underline{v}|0)$.

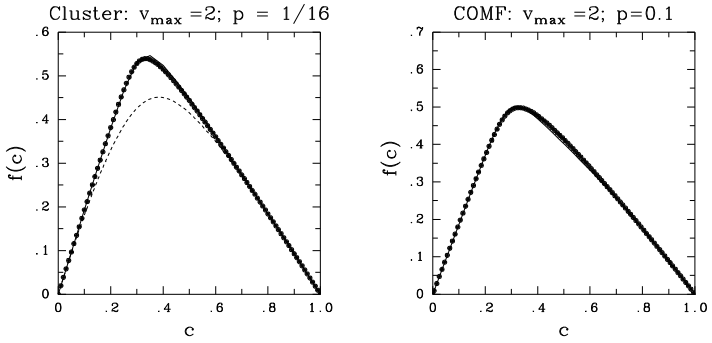

Fig. 8. Fundamental diagram for $v_{max} = 2$: Comparison of simulation data (•) with the 2-cluster ($\cdots$) and 3-cluster results (—) for $p = 1/16$ (left) and with COMF (—) for $p = 0.1$ (right).

The number of jams is given by $\mathcal{N}_J = \sum_{v=1}^{v_{max}} P(\underline{v}|0)$ and one obtains

$$C_1^{(2)} = \frac{1}{\mathcal{N}_J} \sum_{v=1}^{v_{max}} P(\underline{0}|v) P(\underline{v}|0)$$
$$C_n^{(2)} = \frac{1}{\mathcal{N}_J} P(\underline{0}|1) P(\underline{1}|1)^{n-2} P(\underline{1}|1) P(\underline{1}|0) \qquad (n \geq 2). \tag{19}$$

$C_n^{(2)}$ decays exponentially for $n \geq 2$, especially one has $C_{n+1} \leq C_n$. For the m-cluster approximation one can derive similar expressions. Now one finds an exponential decay for jam sizes $n \geq m$, i.e. small clusters may dominate (e.g. $C_3^{(5)} > C_1^{(5)}$ is possible).

In COMF, C_n is proportional to $(1 - P_0)P_0 \cdots P_0(1 - P_0) = (1 - P_0)^2 P_0^{n-1}$. The number of jams is propotional to $1 - P_0$ so that one finds

$$C_n = (1 - P_0) P_0^{n-1}. \tag{20}$$

This distribution is purely exponential and $P_n \geq P_{n+1}$ for all n. Therefore COMF is not able to describe clustering or phase separation, i.e. situations where jams with more than one car dominate. Clustering implies that there are correlations between gaps which are completely neglected in COMF.

Since for $p \to 0$ COMF is exact, there is no tendency towards clustering in this limit. On the hand, for $p \to 1$ COMF fails to give a good description of the phyisics since there is a strong tendency towards clustering or phase separation in that limit. Note that for the (deterministic!) case $p = 1$ the difference between $v_{max} = 1$ and $v_{max} > 1$ becomes most pronounced, since only for $v_{max} > 1$ metastable states exist [15].

8 Other models

In this section we discuss briefly two other models which have been investigated with the methods presented before. Both models are modified NaSch models. The first one is the T^2 model [20,21]. It is a NaSch model with a modified acceleration rule. This so-called 'slow-to-start' rule is designed to mimic the delay of a car in restarting, i.e. due to a slow pick-up of the engine or loss of the driver's attention. We will discuss only the case $v_{max} = 1$. Here the 'slow-to-start' rule has the important effect of breaking the 'particle-hole' symmetry of the NaSch model.

Explicitly the 'slow-to-start' rule is given by:

R1' Acceleration with 'slow-to-start': A standing car will always accelerate to $v = 1$ if there are at least two empty sites in front;
Is there only one empty site, it will accelerate only with probability $1 - p_t$.

Fig. 9 shows the results for the fundamental diagram [21]. Since the 'particle-hole' symmetry is broken, the flow is now maximal at a density $c_{max} < 1/2$. Another interesting effect can be seen for larger values of p_t: The flow vs. density relation has an inflection point and is no longer convex, i.e. the parameter p_t controls the curvature of the fundamental diagram (Fig. 9). This can be seen most clearly for $p_t = 1$. Here the flow $f(c)$ vanishes for $c \geq 1/2$. This new phase transition to a completely jammed state exist only for $p > 0$.

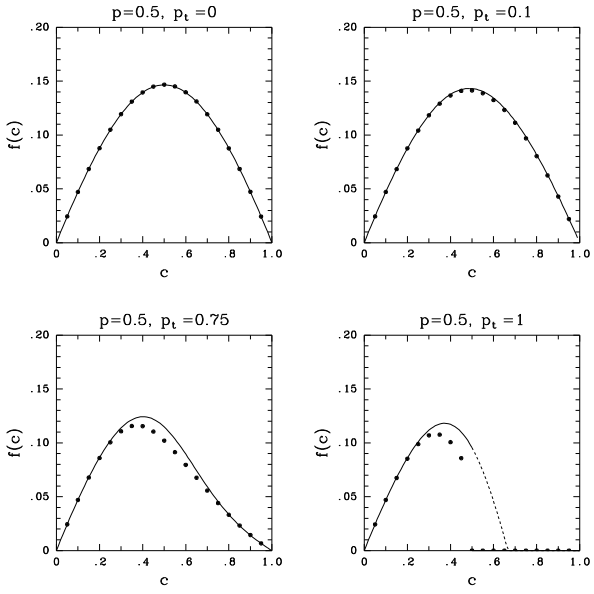

Fig. 9. Fundamental diagram for the T^2 model for different values of p and p_t.

For $c = 1/2$ the stationary state consists of standing cars with exactly one empty site in between them (0.0.0.0.$\cdots$ where '0' denotes a standing car and '.' an empty site). No car can move due to the slow-to-start rule. For larger densities the stationary state is essentially of the same form, but larger clusters of standing cars will appear. In Monte Carlo simulations starting from a megajam the relaxation into this stationary state is extremely slow, even for 'large' values of p, e.g. $p = 0.5$.

The COMF equations have two solutions [21]. Apart from the solution with vanishing flow for $c \geq 1/2$ there exists a second solution with $f(c) > 0$ for $1/2 < c < 2/3$ (dotted line in Fig. 9). Therefore one might expect the existence of a hysteresis effect in this model, which is indeed found in Monte Carlo simulations[4]. This is not surprising since the slow-to-start rule reduces the outflow from a megajam compared to the maximum flow of 'free traffic' which seems to be an important factor for the occurance of hysteresis.

The second model has been introduced by Fukui and Ishibashi [23] and can be considered as a NaSch model for "aggressive driving". Again the rules are identical to the NaSch model, only the acceleration rule is changed:

R1" Acceleration: Every car accelerates to v_{max}.

For $v_{max} = 1$ this does not change anything, but for higher velocities it leads to a considerable enhancement of the flow.

Although the model is less realistic than the NaSch model, it is of interest due to its simplicity. An analytical description is much simpler than that of the NaSch model since after the acceleration step all cars have the same velocity v_{max}. Therefore the analytical description is much simpler and might serve as testing ground for new methods [24].

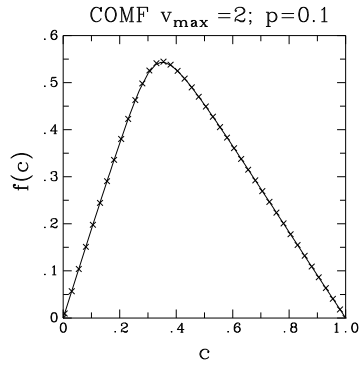

Fig. 10. Fundamental diagram for the Fukui-Ishibashi model: Comparison of simulation results (×) and COMF (—).

[4] A more detailed discussion can be found elsewhere in these proceedings [22].

9 Summary

Although cellular automata are designed for efficient computer simulation studies, an analytical description is possible, although difficult. We have present here four different methods which can be applied to CA models of traffic flow. The first approach, a simple mean-field theory for cell occupation numbers, is insufficient since the important correlations between neighbouring cells (e.g. the particle-hole attraction) are neglected. We therefore suggested three different improved mean-field theories. These approaches take into account certain correlations between the cells. The simplest method is the so-called 'paradisical mean-field' theory which is based on the observation that certain configurations (Garden of Eden states) can never be generated by the dynamical rules due to the use of parallel dynamics. The cluster approximation, on the other hand, treats clusters of a certain size exactly and couples them in a self-consistent way. Therefore short-ranged correlations are taken into account properly. In contrast, car-oriented mean-field theory is a true mean-field theory, but here one uses a different dynamical variable, namely the distance between consecutive cars. In that way, certain correlations between cells are taken into account.

All three improved MF theories become exact for $v_{max} = 1$. For larger values of v_{max} they are just approximations. In principle, the cluster approximation and COMF (in combination with a cluster approach) can be improved systematically. This is, however, very cumbersome.

An interesting observation is that the qualitity of the approximation depends strongly on the value of p. This indicates that the physics changes with p, contrary to common believe. Evidence for this scenario comes from the behaviour of the jam distribution and the phase transition.

The methods presented here can also be used for other CA models. The investigation of the NaSch model has led to a better understanding of their advantages and limitations so that it is easier to choose the approach most suitable for a given problem.

The question, how the stationary state is approached, is still an important open issue. Here, generalizations of the methods presented above can also be applied. This problem is currently under investigation [25] and promises to yield new insights into the physics of the NaSch model.

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